# Interpretation of quantum mechanics by the double solution theory

### Louis de BROGLIE

EDITOR'S NOTE. In this issue of the Annales, we are glad to present an English translation of one of Louis de Broglie's latest articles, as a kind of gift to all physicists abroad who are not well acquainted with the double solution theory, or do not read French. Louis de Broglie of course wrote the original paper[1] in his mother tongue, which he mastered with utmost elegance, but perhaps considering it as his last word on Wave Mechanics, he expressed the wish to see it also published in English.

The translator, our friend Maurice Surdin, tried to remain as close as possible to the French text, which was by no means an easy task, and unavoidably the result will therefore appear a bit awkward in style, but it surely does convey the precise physical meaning, and most importantly, the spirit of Louis de Broglie's work. Following this closeness requirement, the peculiar mathematical notations used by the author have been kept unaltered, even though somewhat unusual, or slightly old-fashioned. Our readers will nevertheless appreciate the deep physical insight expressed in this tentative theory of wave-particle dualism, a major problem unsolved to everyone's satisfaction.

Historically, Einstein was the one who started all the trouble in 1905, with the introduction of this wave-particle dualism in radiation theory. Louis de Broglie did not ease the pressure in theoretical physics when he later on extended the puzzling dualism to every entity of Universe, not only photons, but also electrons, atoms, molecules, etc. And he was right, that is the way things work, and physicists have to accept facts, however upsetting.

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[1] Foundations of Quantum Mechanics - Rendiconti della Scuola Internazionale di Fisica "Enrico Fermi", IL Corso, B. d'Espagnat ed. Academic Press N.Y.1972

But Louis de Broglie, as he explains in the first lines of his article, was a realist, and he could not believe observable physical phenomena to only follow from abstract mathematical wave-functions. Somehow, these latter had to be connected to real waves, at variance with the prevailing Copenhagen interpretation, and with his keen sense for physics, Louis de Broglie did find a way out of the maze !

So here is a realistic view of Wave Mechanics ... at the highest level, and by its very discoverer.

## I. The origin of Wave Mechanics

When in 1923-1924 I had my first ideas about Wave Mechanics [1] I was looking for a truly concrete physical image, valid for all particles, of the wave and particle coexistence discovered by Albert Einstein in his "Theory of light quanta". I had no doubt whatsoever about the physical reality of waves and particles.

To start with, was the following striking remark : in relativity theory, the frequency of a plane monochromatic wave is transformed as $\nu = \nu_0/\sqrt{1 - \beta^2}$ whereas a clock's frequency is transformed according to a different formula : $\nu = \nu_0\sqrt{1 - \beta^2}$ ($\beta = v/c$). I then noticed that the 4-vector defined by the phase gradient of the plane monochromatic wave could be linked to the energy-momentum 4-vector of a particle by introducing $h$, in accordance with Planck's ideas, and by writing :

$$W = h\nu \qquad p = h/\lambda \tag{1}$$

where $W$ is the energy at frequency $\nu$, $p$ the momentum, and $\lambda$ the wavelength. I was thus led to represent the particle as constantly localized at a point of the plane monochromatic wave of energy $W$, momentum $p$, and moving along one of the rectilinear rays of the wave.

However, and this is never recalled in the usual treatises on Wave Mechanics, I also noticed that if the particle is considered as containing a rest energy $M_0c^2 = h\nu_0$ it was natural to compare it to a small clock of frequency $\nu_0$ so that when moving with velocity $v = \beta c$, its frequency different from that of the wave, is $\nu = \nu_0\sqrt{1 - \beta^2}$. I had then easily shown that while moving in the wave, the particle had an internal vibration which was constantly in phase with that of the wave.

The presentation given in my thesis had the drawback of only applying to the particular case of a plane monochromatic wave, which

is never strictly the case in nature, due to the inevitable existence of some spectral width. I knew that if the complex wave is represented by a Fourier integral, i.e. by a superposition of components, these latter only exist in the theoretician's mind, and that as long as they are not separated by a physical process which destroys the initial superposition, *the superposition is the physical reality*. Just after submitting my thesis, I therefore had to generalize the guiding ideas by considering, on one hand, a wave which would not be plane monochromatic, and on the other hand, by making a distinction between the real physical wave of my theory and the fictitious $\psi$ wave of statistical significance, which was arbitrarily normed, and which following Schrödinger and Bohr's works was starting to be systematically introduced in the presentation of Wave Mechanics. My arguments were presented in Journ. de Phys. May 1927 [2], and titled : "The double solution theory, a new interpretation of Wave Mechanics". It contained a generalization of a particle's motion law for the case of any wave ; this generalization was not considered at the start in the particular case of a plane monochromatic wave.

Contemplating the success of Quantum Mechanics as it was developed with the Copenhagen School's concepts, I did for some time abandon my 1927 conceptions. During the last twenty years however, I have resumed and greatly developed the theory.

## II. The double solution theory and the guidance rule

I cannot review here in detail the present state of the double solution theory. A complete presentation may be found in the referenced publications. However I would like to insist on the two main and basic ideas of this interpretation of Wave Mechanics. A/- In my view, the wave is a physical one having a very small amplitude which cannot be arbitrarily normed, and which is distinct from the $\psi$ wave. The latter is normed and has a statistical significance in the usual quantum mechanical formalism. Let $v$ denote this physical wave, which will be connected with the statistical $\psi$ wave by the relation $\psi = Cv$, where $C$ is a normalizing factor. The $\psi$ wave has the nature of a subjective probability representation formulated by means of the objective $v$ wave. This distinction, essential in my opinion, was the reason for my naming the theory "Double solution theory", for $v$ and $\psi$ are thus the two solutions of the same wave equation. B/- For me, the particle, precisely located in space at every instant, forms on the $v$ wave a small region of high energy concentration, which may be likened in a first approximation,

to a moving singularity. Considerations which will be developed further on, lead to assume the following definition for the particle's motion : if the complete solution of the equation representing the $v$ wave (or if prefered, the $\psi$ wave, since both waves are equivalent according to the $\psi = Cv$ relation) is written as :

$$v = a(x, y, z, t).exp\left(\frac{i}{\hbar}\, \phi(x, y, z, t)\right) \qquad \hbar = h/2\pi \qquad (2)$$

where $a$ and $\phi$ are real functions, energy $W$ and momentum $p$ of the particle, localized at point $x, y, z$, at time $t$, are given by :

$$W = \frac{\partial \phi}{\partial t} \qquad \vec{p} = -\overrightarrow{\text{grad}}\phi \qquad (3)$$

which in the case of a plane monochromatic wave, where one has

$$\phi = h\left(\nu t - \frac{\alpha x + \beta y + \gamma z}{\lambda}\right)$$

yields eq. (1) for $W$ and $p$.

If in eq. (3) $W$ and $p$ are given as

$$W = \frac{M_0 c^2}{\sqrt{1 - \beta^2}} \qquad \vec{p} = \frac{M_0 \vec{v}}{\sqrt{1 - \beta^2}}$$

one gets :

$$\vec{v} = \frac{c^2 \vec{p}}{W} = -c^2\, \frac{\overrightarrow{\text{grad}}\,\phi}{\partial \phi/\partial t} \qquad (4)$$

I called this relation, which determines the particle's motion in the wave, "the guidance formula". It may easily be generalized to the case of an external field acting on the particle.

Now, going back to the origin of Wave Mechanics, will be introduced the idea according to which the particle can be likened to a small clock of frequency $\nu_0 = M_0 c^2/h$, and to which is given the velocity of eq. (4). For an observer seeing the particle move on its wave with velocity $\beta c$, the internal frequency of the clock is $\nu = \nu_0\sqrt{1 - \beta^2}$ according to the relativistic slowing down of moving clocks. As will be shown further on, it is easily demonstrated that in the general case of a wave which is not plane monochromatic, the particle's internal vibration is constantly in

phase with the wave on which it is carried. This result, including as a particular case that of the plane monochromatic wave first obtained, can be considered the main point of the guidance law.

As will be seen further on, it can easily be shown that the proper mass $M_0$ which enters the relation giving $M$ and $p$ is generally not equal to the proper mass $m_0$ usually given to the particle. One has :

$$M_0 = m_0 + Q_0/c^2 \tag{5}$$

where, in the particle's rest frame, $Q_0$ is a positive or negative variation of the rest mass. The quantity $Q_0$ is the "quantum potential" of the double solution theory. Its dependence on the variation of the wave function's amplitude will be seen.

## III. Further study of the double solution theory

Following the sketch of the double solution theory considered above, its fundamental equations will be hereafter developed starting with Schrödinger and Klein-Gordon's wave equations, i.e. without introducing spin. The extension of what follows to spin 1/2 particles as the electron, and to spin 1 particles as the photon, may be found in books $(3a)$ and $(3b)$. The study will be limited to the case of the $v$ wave following the non-relativistic Schrödinger equation, or the relativistic Klein-Gordon equation, which for the Newtonian approximation $(c \to \infty)$ degenerates to the Schrödinger equation.

It is well known that an approximate representation of the wave properties of the electron is obtained in this way.

First taking Schrödinger's equation for the $v$ wave, $U$ being the external potential, one gets :

$$\frac{\partial v}{\partial t} = \frac{\hbar}{2im} \Delta v + \frac{i}{\hbar} U.v \tag{6}$$

This complex equation implies that the $v$ wave is represented by two real functions linked by the two real equations, which leads to :

$$v = a.\exp(i\phi/\hbar) \tag{7}$$

where $a$ the wave's amplitude, and $\phi$ its phase, are real. Taking this value into eq. (6), readily gives :

$$\frac{\partial \phi}{\partial t} - U - \frac{1}{2m} \left(\overrightarrow{\text{grad}\,\phi}\right)^2 = -\frac{\hbar^2}{2m}.\frac{\Delta a}{a} \tag{J}$$

$$\frac{\partial (a^2)}{\partial t} - \frac{1}{m} \cdot \mathrm{div}(a^2 \overrightarrow{\mathrm{grad}} \phi) = 0 \qquad (C)$$

For reasons which will further on become clear, equation $(J)$ will be called "Jacobi's generalized equation", and equation $(C)$ the "continuity equation".

In order to get a relativistic form of the theory, Klein-Gordon's equation is used for the $v$ wave, which instead of eq. (6) gives :

$$\Box v - \frac{2i}{\hbar} \cdot \frac{eV}{c^2} \cdot \frac{\partial v}{\partial t} + \frac{2i}{\hbar} \frac{e}{c} \sum_{xyz} A_x \frac{\partial v}{\partial x} + \frac{1}{\hbar^2} \left( m_0^2 c^2 - \frac{e^2}{c^2}(V^2 - A^2) \right) v = 0 \;\; (8)$$

where it is assumed that the particle has electric charge $e$ and is acted upon by an external electromagnetic field with scalar potential $V(x, y, z, t)$ and vector potential $\vec{A}(x, y, z, t)$.

Insertion of eq. (7) into eq. (8) gives a generalized Jacobi equation $(J')$ and a continuity equation $(C')$ as follows :

$$\frac{1}{c^2} \left( \frac{\partial \phi}{\partial t} - eV \right)^2 - \sum_{xyz} \left( \frac{\partial \phi}{\partial x} + \frac{e}{c} A_x \right)^2 = m_0^2 c^2 + \hbar^2 \frac{\Box a}{a} = M_0^2 c^2 \quad (J')$$

$$\frac{1}{c^2} \left( \frac{\partial \phi}{\partial t} - eV \right) \frac{\partial a}{\partial t} - \sum_{xyz} \left( \frac{\partial \phi}{\partial x} + \frac{e}{c} A_x \right) \frac{\partial a}{\partial x} + \frac{a}{2} \Box \phi = 0 \qquad (C')$$

where on the right hand side of $(J')$ was introduced a variable proper mass $M_0$ which is defined by :

$$M_0 = \left( m_0^2 + (\hbar^2/c^2) \frac{\Box a}{a} \right)^{1/2} \qquad (9)$$

this quantity, as will be seen further, is of great importance.

## IV. The guidance formula and the quantum potential

Let us now consider equations $(J)$ and $(J')$ corresponding to the non-relativistic Schrödinger, and relativistic Klein-Gordon equations.

First taking Schrödinger's equation and eq. $(J)$, if terms involving Planck's constant $h$ are neglected on the right hand side, which amounts to disregard quanta, and if $\phi$ is set as $\phi = S$, then eq. $(J)$ becomes :

$$\frac{\partial S}{\partial t} - U = \frac{1}{2m} (\overrightarrow{\mathrm{grad}} \, S)^2 \qquad (10)$$

As $S$ is the Jacobi function, eq. (10) is the Jacobi equation of classical mechanics. This means that only the term with $\hbar^2$ is responsible for the particle's motion being different from the classical motion. What is the significance of this term? It may be interpreted as another potential $Q$, distinct from the classical $U$ potential, $Q$ being given as :

$$Q = -\frac{\hbar^2}{2m}.\frac{\Delta a}{a} \tag{11}$$

By analogy with the classical formulae $\partial S/\partial t = E$, and $\vec{p} = -\overrightarrow{\text{grad}}S$, $E$ and $p$ being the classical energy and momentum, one may write :

$$\frac{\partial \phi}{\partial t} = E \qquad -\overrightarrow{\text{grad}}\phi = \vec{p} \tag{12}$$

As in non-relativistic mechanics, where $p$ is expressed as a function of velocity by the relation $\vec{p} = m.\vec{v}$, the following is obtained :

$$\vec{v} = \vec{p}/m = -\frac{1}{m}\overrightarrow{\text{grad}}\phi \tag{13}$$

This equation is called the "guidance formula"; it gives the particle's velocity, at position $x, y, z$, and time $t$, as a function of the local phase variation at this point.

It should be stressed that $a$ and $\phi$, the amplitude and phase of the $v$ wave, would exist if a minute region of very high amplitude, which is the particle, did not itself exist. At one's preference, it may be said that $a$ and $\phi$ are the amplitude and phase of the $v$ wave, in direct proximity of the pointlike region $u_0$, of a wave defined by $u = u_0 + v$. I gave justifications of the guidance formula, based on this idea. This problem will be reconsidered further on.

The quantum force $\vec{F} = -\overrightarrow{\text{grad}}Q$ acting on the particle, bends its trajectory. However, in the important albeit schematic case of a plane monochromatic wave, $Q$ is constantly zero, and there is no quantum force; the particle moves with constant velocity along a rectilinear trajectory. This latter is one of the plane monochromatic wave's rays; the image I had in mind while writing my thesis is thus found again.

However, when the wave's propagation is subject to boundary conditions, interference or diffraction phenomena do appear; owing to

the quantum force, the motion defined by the guidance formula is not rectilinear any more. It then happens that the obstacles hindering the propagation of the wave act on the particle through the quantum potential, in this way producing a deflection. Supporters of the ancient "emission theory" thought that light was exclusively formed of particles, and as they already knew that light may skirt around the edge of a screen, they considered this edge as exerting a force on the light particles which happened to pass in its neighbourhood. Under a more elaborate form, here again we find a similar idea.

Let us now consider Klein-Gordon's equation and eq. $(J')$.

It may first be noticed, that neglecting terms in $\hbar^2$ in eq. $(J')$ gives :

$$\frac{1}{c^2}\left(\frac{\partial S}{\partial t} - eV\right)^2 - \sum_{xyz}\left(\frac{\partial S}{\partial x} + eA_x\right)^2 = m_0^2 c^2 \tag{14}$$

As should be expected in relativistic mechanics without quanta, this equation is Jacobi's equation for a particle of proper mass $m_0$ and electric charge $e$, moving in an electromagnetic field with scalar and vector potentials respectively $V$ and $\vec{A}$. Keeping the terms in $\hbar^2$ and considering the proper mass $M_0$ as defined in eq. (9) naturally leads to :

$$\frac{M_0 c^2}{\sqrt{1-\beta^2}} = \frac{\partial \phi}{\partial t} - eV \qquad \frac{M_0 \vec{v}}{\sqrt{1-\beta^2}} = -(\overrightarrow{\mathrm{grad}}\phi + e\vec{A}) \tag{15}$$

with $\beta = v/c$, which in turn leads to the relativistic guidance formula :

$$\vec{v} = -c^2 \frac{\overrightarrow{\mathrm{grad}}\phi + e\vec{A}}{\partial\phi/\partial t - eV} \tag{16}$$

For the Newtonian approximation, with $A = 0$ and $\frac{\partial\phi}{\partial t} - eV \cong m_0 c^2$, eq. (13) is obtained as it should.

Here, the quantum force results from the variation of the $M_0 c^2$ quantity, as the particle moves in its wave. In the case of a plane monochromatic wave, for the quantum potential to be constantly zero, one writes :

$$Q = M_0 c^2 - m_0 c^2 \tag{17}$$

For the non-relativistic approximation, with $c \to \infty$ and $\bar{\Box} a \cong -\Delta a$, the following value is reached :

$$Q = \sqrt{\frac{m_0^2 c^4 + c^2 \hbar^2 \, \Box a}{a}} - m_0 c^2 \cong -\frac{\hbar^2}{2m_0} . \frac{\Delta a}{a}$$

The above relativistic relations are most important for what follows, because Wave Mechanics is an essentially relativistic theory, as I perceived at its beginning ; Schrödinger's equation, being non-relativistic, is improper to reveal its true nature.

## V. Interpretation of the motion guidance

Two important characteristics of the motion guidance will now be stressed. The first one is that the particle moving on its wave, is essentially in phase with it. To prove this, suppose first that no other than the quantum force acts on the particle, which is equivalent to making $V = A = 0$ in the Klein-Gordon equation. If one moves along the guiding trajectory by a distance $dl$ in time $dt$, the corresponding phase variation of the wave is :

$$d\phi = \frac{\partial \phi}{\partial t} \, dt + \frac{\partial \phi}{\partial l} \, dl = \left( \frac{\partial \phi}{\partial t} + \vec{v}.\overrightarrow{\text{grad}}\phi \right) dt$$

$$d\phi = \left( \frac{M_0 c^2}{\sqrt{1 - \beta^2}} - \frac{M_0 v^2}{\sqrt{1 - \beta^2}} \right) dt = (M_0 c^2 \sqrt{1 - \beta^2}) dt \qquad (18)$$

When the particle of internal frequency $\nu_0 = M_0 c^2 / h$, has moved a distance $dl$ during $dt$, its internal phase $\phi_i$ has changed by :

$$d\phi_i = M_0 c^2 \sqrt{1 - \beta^2} . dt = d\phi \qquad (19)$$

The particle when in motion on its wave, thus has its vibration [2] constantly in phase with that of the wave.

This result may be interpreted by noticing that, in the present theory, the particle is defined as a very small region of the wave where the amplitude is very large, and it therefore seems quite natural that the

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[2]   defined by $a_i \, exp(i.\phi_i/\hbar)$ with $a_i$ and $\phi_i$ real.

internal motion rythm of the particle should always be the same as that of the wave at the point where the particle is located.

A very important point must be underlined here. For this interpretation of the guidance to be acceptable, the dimensions of the minute singular region constituting the particle ought to be very small compared to the wavelength of the $v$ wave. It might be considered that the whole theory has its validity limited to very short wavelengths, i.e. very high energies. This remark has little importance for usually considered cases, but may become of primary importance for very high energies.

The foregoing demonstration may be extended to the case in which $V$ and $A$ are not zero in Klein-Gordon's equation. The phase concordance of wave and particle is still expressed by :

$$\left(\frac{\partial \phi}{\partial t} + \vec{v}.\overrightarrow{\text{grad}}\phi\right) dt = \frac{d\phi_i}{dt}.dt \tag{20}$$

Let $h\nu_0 = \partial\phi/\partial t_0 = M_0 c^2 + eV_0 = M_0' c^2$, and thus, $M_0' c^2 = M_0 c^2 + eV_0$ in the particle's proper frame, where it is momentarily at rest. On the other hand :

$$\frac{d\phi}{dt} = h\nu = \frac{h\nu_0}{\sqrt{1-\beta^2}} = \frac{M_0' c^2}{\sqrt{1-\beta^2}}$$

$$\frac{\partial \phi}{\partial t} = h\nu_i = h\nu_0 \sqrt{1-\beta^2} = M_0' c^2 \sqrt{1-\beta^2} \tag{21}$$

eq. (20) is therefore obtained.

There is another characteristic of the guided motion. *The motion is performed according to relativistic dynamics of a variable proper mass.* To prove this, in absence of classical fields, the following Lagrange function is considered

$$\mathcal{L} = -M_0 c^2 \sqrt{1-\beta^2} \tag{22}$$

The least action principle $\delta \int \mathcal{L}\, dt = 0$ yields the Lagrange equations :

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right) = \frac{\partial \mathcal{L}}{\partial q_i} \tag{23}$$

which in the present case is :

$$\frac{d\vec{p}}{dt} = -c^2 \sqrt{1-\beta^2}\, \overrightarrow{\text{grad}} M_0 \tag{24}$$

This shows that the particle obeys relativistic dynamics of a variable proper mass. With the relativistic symmetry between space and time, eq. (24) may be complemented by :

$$\frac{dW}{dt} = c^2\sqrt{1-\beta^2}.\frac{\partial M_0}{\partial t} \qquad (25)$$

and as $dM_0/dt = \partial M_0/\partial t + \vec{v}.\overrightarrow{\mathrm{grad}}\,M_0$, the preceding equations give :

$$\frac{dW}{dt} - \vec{v}.\frac{\partial \vec{p}}{dt} = c^2\sqrt{1-\beta^2}.\frac{dM_0}{dt} \qquad (26)$$

Keeping in mind that :

$$\vec{v}.\frac{d\vec{p}}{dt} = \frac{d(\vec{v}.\vec{p})}{dt} - \vec{p}.\frac{d\vec{v}}{dt} = \frac{d}{dt}(\vec{v}.\vec{p}) - \frac{M_0\vec{v}}{\sqrt{1-\beta^2}}.\frac{d\vec{v}}{dt}$$

$$c^2\sqrt{1-\beta^2}\,\frac{dM_0}{dt} = \frac{d}{dt}\left(M_0 c^2\sqrt{1-\beta^2}\right) + \frac{M_0\vec{v}}{\sqrt{1-\beta^2}}.\frac{d\vec{v}}{dt} \qquad (27)$$

one obtains :

$$\frac{d}{dt}(W - \vec{v}.\vec{p} - M_0 c^2\sqrt{1-\beta^2}) = 0 \qquad (28)$$

and as it was assumed that when the particle is at rest, $\beta = 0$, and $W = M_0 c^2$, there comes :

$$W = M_0 c^2\sqrt{1-\beta^2} + \vec{v}.\vec{p} = M_0 c^2\sqrt{1-\beta^2} + \frac{M_0 v^2}{\sqrt{1-\beta^2}} \qquad (29)$$

This relation, obtained from the guidance dynamics of variable proper mass, is verified since $W = M_0 c^2/\sqrt{1-\beta^2}$, and as will be seen, has a remarkable thermodynamical significance.

The preceding argument may be generalized for the case where there is an external electromagnetic field, by using the following Lagrange function :

$$\mathcal{L} = -M_0 c^2\sqrt{1-\beta^2} + e(V - \vec{A}.\vec{v}/c) = M_0' c^2\sqrt{1-\beta^2} \qquad (30)$$

where the relativistic transformation $V_0 = \frac{V-\vec{v}.\vec{A}/c}{\sqrt{1-\beta^2}}$ was used.

## VI. Interpretation of the continuity equations $(C)$ and $(C')$

Let us consider the significance of equations $(C)$ and $(C')$, formely derived in §III, and respectively corresponding to the non-relativistic Schrödinger, and relativistic Klein-Gordon equations.

First considering eq. $(C)$

$$\frac{\partial a^2}{\partial t} - \frac{1}{m} \cdot \operatorname{div}(a^2 \overrightarrow{\operatorname{grad}} \phi) = 0 \qquad (C)$$

using the guidance formula (4), and setting $\rho = Ka^2$, where $K$ is a constant, eq. $(C)$ becomes :

$$\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho \vec{v}) = 0 \qquad (31)$$

In hydrodynamics, this equation is called the continuity equation. $\rho d\tau$ is the number of the fluid's molecules in the volume element $d\tau$, and $\vec{v}$ is the velocity. This equation may be written as $D(\rho d\tau)/Dt = 0$, where the $D/Dt$ derivative is taken along the motion of the molecules ; it expresses the conservation of the fluid. In the present case however, there is a single particule, and it seems quite natural to consider the quantity $\rho d\tau$ as proportional to the probability of finding the particle in the elementary volume $d\tau$. As will be shown further on, this interpretation raises a problem if one assumes that the particle regularly follows its guided trajectory, and this difficulty leads to complementing the guidance theory, as it was developed above, by introducing a random element, which will open up new vistas.

Without further insisting on this point, it is assumed that $\rho = a^2(x, y, z, t)$ multiplied by $d\tau$ gives, with a normalizing factor, the probability of finding the particle at time $t$, in the volume element $d\tau$, located at $x, y, z$. We will have to define the statistical function $\psi$ as a function of the real $v$ wave by the relation $\psi = Cv$, with $C$ a normalizing factor, and such that $\int |\psi|^2 d\tau = 1$, so we are led to saying that $|\psi|^2 d\tau$ represents the considered probability's absolute value of finding the particle in $d\tau$.

Let us now examine eq. $(C')$ which corresponds to the relativistic Klein-Gordon equation. Multiplication by $2a$ yields :

$$\frac{1}{c^2} \left( \frac{\partial \phi}{\partial t} - eV \right) \frac{\partial a^2}{\partial t} - (\overrightarrow{\operatorname{grad}}\phi + e.A/c)\overrightarrow{\operatorname{grad}}a^2 + a^2 + a^2 \Box \phi = 0 \quad (C')$$

$\rho$ will here be defined by :

$$\rho = Ka^2 \left( \frac{\partial \phi}{\partial t} - eV \right) \qquad (32)$$

With this definition, and by use of eq. (16), which gives the guiding velocity, and of the Lorentzian relation between potentials $\frac{1}{c}\frac{\partial V}{\partial t} +$ div $\vec{A} = 0$, the continuity relation (31) is found again.

It may be assumed, with the same meaning as before, that whenever the relativistic Klein-Gordon wave equation is used, the quantity

$$\rho d\tau = Ka^2 \left( \frac{\partial \phi}{\partial t} - eV \right) d\tau$$

gives the probability of finding the particle in the volume element $d\tau$ at time $t$.

## VII. Introduction of the statistical $\psi$ wave

Above, was only introduced the $v$ wave, containing a very small singular region, generally in motion, which constitutes the particle. This very small amplitude $v$ wave, which is spread out over almost all the domain occupied by the $u$ wave, $u = u_0 + v$, $u_0$ here representing the high amplitude wave in this minute region, may be quite complex.

It defines the particle's internal structure. We will not insist on this point the study of which at the time being seems premature. It looks quite natural that the propagation in space and time of the truly physical $v$ wave should determine, as was assumed in the guidance theory, the particle's motion, as it is integrated into the wave. Just as well, the stationary states of the $v$ wave, in systems such as the Hydrogen atom, could determine the quantum states of that system.

However, since the publication of Schrödinger's works in 1926, it became customary to only consider the $\psi$ wave, of arbitrarily normed amplitude. But this wave cannot be considered as a physical wave, first because the amplitude of a physical wave has a well determined value, and cannot be arbitrarily normed, and because if $\psi_1$ and $\psi_2$ are two particular normed solutions of the linear $\psi$ wave, the $\psi_1 + \psi_2$ sum of these two solutions is not a normed solution. In other words, the normed $\psi$ wave is not endowed with the superposition property

characteristic of the physical wave solutions of a linear propagation equation. One is therefore led to consider the $\psi$ wave as a probability representation, a simple prediction instrument, permitting a forecast of the possible measurement results of physical quantities belonging to a particle or to an ensemble of particles. It is however impossible for a simple probability representation to create physical phenomena such as the local observation of a particle, or to impose definite values to energies of atomic stationary states. Objective reality only, may give such effects, and a probability representation has no such character.

It is nevertheless unquestionable that use of the $\psi$ wave and its generalization did lead to accurate prediction and fruitful theories. This is an indisputable fact. The situation is clarified by introducing together with the statistical $\psi$ wave, the $v$ wave, which being an objective physical reality, may give rise to phenomena the statistical aspect of which is given by the $\psi$ wave. It becomes important to establish the relationship between the $\psi$ and $v$ waves.

By means of a constant $C$, which may be complex, the required relation is introduced as :

$$\psi = C.v = C.a.\exp(i\phi/\hbar) \tag{33}$$

$C$ is a normalizing factor such that $\int_V \mid \psi \mid^2 \ d\tau = 1$, $V$ denoting the volume occupied by the $v$ wave. As seen in the preceding section, where in Schrödinger's theory, $\mid \psi \mid^2 \ d\tau$ gave the probability of finding the particle in the volume element $d\tau$, the quantity $\mid \psi \mid^2 \ d\tau$ gives the *absolute value* of the probability, and this is the reason for introducing the normed statistical $\psi$ function in relation with the real $v$ wave of eq. (33).

One first remark about eq. (33) is the following : as $\mid \psi \mid = \mid C \mid a$, and as the phase of $\psi$ cannot be different from that of $v$ but for an additive constant, the guidance formulae and the expression giving the quantum potential previously considered are indifferent to the substitution of $v$ by $\psi$.

Another remark is that $\mid C \mid$ ought to be much larger than 1. Consider a quantity related to the particle whose value $g$ is known. The current theory which only uses the $\psi$ function, assumes this quantity to be spread out over the whole wave with density $\mid \psi \mid^2 \ d\tau$ so that $\int \mid \psi \mid^2 \ d\tau = g$. In the double solution theory however, the quantity $g$ is certainly concentrated in a very small region occupied by the particle,

and the integral of $a^2 g \, d\tau$ taken over the $v$ wave in the volume $V$ is much smaller than $g$, so that :

$$\int_V a^2 g d\tau \ll \int_V \mid \psi \mid^2 \ g d\tau \tag{34}$$

which by use of eq. (33) gives :

$$\mid C \mid \gg 1 \tag{35}$$

This result may be interpretated by stating that the current statistical theory considers as spread out in the entire wave, devoid of singularity, that which in reality is totally concentrated in the singularity. It is on account of the foregoing interpretation that I simultaneously considered two distinct solutions of the wave propagation equation connected by eq. (33), one, $v$, having physical reality, and the other, $\psi$, normed, and of statistical character. I therefore named this reinterpretation of wave mechanics the double solution theory. By distinction of the two waves $v$ and $\psi$, the mystery of the double character, subjective and objective, of the wave in the usual theory, vanishes, and one no longer has to give a simple probability representation the strange property of creating observable phenomena.

Moreover, the distinction between the $v$ and $\psi$ waves leads to a new outlook on a large number of important problems such as the interpretation of interference phenomena, measurement theory, distant correlations, definition of pure and mixed states, reduction of a probability wave packet, etc. The results obtained during the last few years by Mr. Andrade e Silva and myself, show improved clarity and accuracy compared to the presently used theories. Without further insistence, it should be noted that Mr. Andrade e Silva has recently considered pure and mixed states, so defining the corresponding statistical function $\psi = Cv$, which in some cases differs from the usual $\psi$ function.

## VIII. Localization of the particle in the wave and the guidance law

Thusfar, the insertion of the particle in its wave was restrictively defined by stating that the real physical wave must include a small region of very high amplitude, which is the particle. Apart from this singular region, the physical wave is the $v$ wave, of very limited amplitude,

and satisfying the usual linear equation. As previously stated, it seems premature to try and describe the internal structure of this singular region, i.e. the particle. This description will probably involve complicated non-linear equations.

The problem that may be considered with some confidence, is the justification of the guidance law, by examining how the singular region should move in the regular wave surrounding it. Some years ago, I did present arguments justifying the guidance law [3]. These are essentially based on the way in which quantities respectively characterizing the regular $v$ wave and the internal $u_0$ wave of the particle connect with the neighbourhood of the singular region. $u_0$ would have to increase very sharply as one penetrates the singular region.

These arguments present great similarity with those used by Einstein and his co-workers to justify in General Relativity the statement that a material particle moves along a space-time geodesic. This problem, which concerned Einstein, has received a thorough solution from Darmois and Lichnerowicz. Their demonstration is based on the consideration of a slender tube of Universe which in space-time represents the particle's motion. They further say that there is a continuous link at the periphery of the tube between the inside and the outside geodesics. Transposing this method to the double solution theory, it may be said that the particle moves in the internal space of a very slender tube, the walls of which are made up by an ensemble of the $v$ wave's stream lines, so defining a hydrodynamical flow. As these stream lines follow from the velocity $v$ of the guidance theory, the particle remains inside the tube during its motion, and the guidance law of the particle by the $v$ wave results. In spite of the fact that the physical nature of the problems in general relativity and double solution theory are different, the methods of demonstration are the same.

Another more schematic way of approaching the problem exists however. The particle is represented as a mathematical singularity inside the wave, and a solution to the wave equation, with moving singularity, is looked for. I gave an outline of this method in my Journ. de Phys. 1927 article [2]. I then showed that using Klein-Gordon's equation, solutions might be found, having the phase of a plane monochromatic wave and a mobile singularity. It was important to generalize this result beyond the particular case I have considered. The problem was studied by Francis

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[3]  See Ref. (3a) Chap. IX p. 101 and appendix p. 287.

Fer in his doctorate thesis, and further extended by Thiounn in a series of articles [4]. Thiounn has shown that in the case of Klein-Gordon's equation applying to 0 spin particles, as well as in the case of Dirac's equation applying to spin $\hbar/2$ particles (electrons in particular), and also in the case of Maxwell's equation with terms representing mass and applying to spin $\hbar$ particles (photons particularly), solutions exist having a pointlike singularity moving according to the guidance law. Representation of a particle by a simple singularity moving along the wave is surely not a true picture of the particle's structure, but only a very schematic one. However, I consider Thiounn's work as very important, and as a remarkable confirmation of the guidance theory.

### IX. The hidden thermodynamics of particles

I will now present the main ideas of the hidden thermodynamics of particles, which I developed since 1960 [5] as an extension of the double solution theory.

The idea of considering the particle as a small clock naturally leads to look at the self energy $M_0 c^2$ as the hidden heat of the particle. From this point of view, a small clock has in its proper system an internal periodic energy of agitation which does not contribute to momentum of the whole. This energy is similar to that of a heat-containing body in an internal state of equilibrium.

The relativistic transformation formula for heat, known since Planck and von Laue's works circa 1908, will be used here. If the heat content of a body, in internal homogeneous equilibrium, is $Q_0$ in its rest frame, in another frame where the body has an ensemble velocity $\beta c$, the contained heat becomes :

$$Q = Q_0 \sqrt{1 - \beta^2} \qquad (36)$$

Although this formula, unquestioned for a long time, was recently challenged, I have, in recent years, become firmly convinced that it is accurate [4], and certainly applies to the case of a small body such as a particle. Therefore, if a particle contains in its proper frame, a quantity of heat $Q_0 = M_0 c^2$, the heat quantity it carries in a frame in which it has velocity $\beta c$ will be :

$$Q = Q_0 \sqrt{1 - \beta^2} = M_0 c^2 \sqrt{1 - \beta^2} = h\nu_0 \sqrt{1 - \beta^2} \qquad (37)$$

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[4] See ref. (5b), (5c), and (5d).

The particle thus appears as being at the same time both a small clock of frequency $\nu = \nu_0 \sqrt{1 - \beta^2}$ and the small reservoir of heat $Q = Q_0 \sqrt{1 - \beta^2}$, moving with velocity $\beta c$. This identity of relativistic transformation formulae for a clock's frequency and for heat, does make the double aspect possible.

When the particle moves according to the guidance law, if the wave is not plane monochromatic, the proper mass $M_0$ varies according to eq. (9), if the expression for the wave is known. As previously seen, the particle's motion is governed by relativistic dynamics of a body with variable proper mass, and this suggests a close relation between the fundamental formula of relativistic thermodynamics and the guidance formula. It results from the following argument.

If $\phi$ is the wave's phase, given by $a. \exp(i\phi/\hbar)$, where $a$ and $\phi$ are real, the guidance theory states that :

$$\frac{\partial \phi}{\partial t} = \frac{M_0 c^2}{\sqrt{1-\beta^2}}$$

$$- \overrightarrow{\mathrm{grad}}\, \phi = \frac{M_0 \vec{v}}{\sqrt{1 - \beta^2}} \tag{38}$$

On the other hand the Planck-Laue eq. (37) may be written :

$$Q = M_0 c^2 \sqrt{1 - \beta^2} = \frac{M_0 c^2}{\sqrt{1 - \beta^2}} - \vec{v}.\vec{p} \tag{39}$$

Combination of (38) and (39) then gives :

$$M_0 c^2 \sqrt{1 - \beta^2} = \frac{\partial \phi}{\partial t} + \vec{v}. \overrightarrow{\mathrm{grad}}\, \phi = \frac{d\phi}{dt} \tag{40}$$

but since the particle is likened to a clock of proper frequency $M_0 c^2/h$, the phase of its internal vibration, written as $a_i \exp(i.\phi_i/\hbar)$, with $a_i$ and $\phi_i$ real, is :

$$\phi_i = h\nu_0 \sqrt{1 - \beta^2}.t = M_0 c^2 \sqrt{1 - \beta^2}.t \tag{41}$$

and therefore :

$$d(\phi_i - \phi) = 0 \tag{42}$$

This agrees with the fundamental assumption according to which the particle as it moves in its wave, remains constantly in phase with it [5]. Thus, there exists a close relation between the guidance theory and relativistic thermodynamics. This fact is even more remarkable when one thinks that eq. (36), the Planck and Laue result, dates back many decades before the emergence of Wave mechanics and the double solution theory.

## X. The relation between action and entropy

Following the above arguments, another one seems natural. Relativistic dynamics states that the Lagrangian of a free particle with proper mass $M_0$, and velocity $\beta c$ is $\mathcal{L} = -M_0 c^2 \sqrt{1 - \beta^2}$, and that :

$$\int \mathcal{L} dt = - \int M_0 c^2 \sqrt{1 - \beta^2} . dt \tag{43}$$

is the action integral, an invariant quantity since $M_0 c^2 \sqrt{1 - \beta^2} . dt = M_0 c^2 . dt_0$, where $dt_0$ is the particle's proper time element. In agreement with an idea of Eddington's some fifty years back, it is tempting to try and establish a relation between the two major "invariants" of physics, Action and Entropy. This however is only possible by giving the action integral of eq. (43) a well defined value by a proper choice of the integration interval. Following the preceding ideas, it is natural to choose as this integration interval the period $T_i$ of internal vibration of the particle with proper mass $m_0$, in the reference frame where its velocity is $\beta c$. Since $1/T_i = (m_O c^2 / h) \sqrt{1 - \beta^2}$, a "cyclic" action integral is defined by noticing that the $T_i$ period is always very short, and therefore $M_0$ and $\beta$ may be considered as practically constant during the integration interval. Then defining action $A$ by

$$A/h = - \int_0^{T_i} M_0 c^2 \sqrt{1 - \beta^2} . dt = - \frac{M_0 c^2}{m_0 c^2} \tag{44}$$

and denoting the hidden thermostat's entropy by $S$, there comes :

$$S/k = A/h \tag{45}$$

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[5] This assumption is only valid for a fairly massive particle, so that it would not undergo a Brownian motion due to energy exchange with the sub-quantum medium. It is not valid in the case of an electron on account of its too low mass.

where $k$ and $h$ are respectively the Boltzmann and Planck constants. Since $\delta Q_0 = \delta M_0 c^2$, it follows that :

$$\delta S = -k\delta Q_0/(m_0 c^2) \tag{46}$$

An entropy has thus been given to the particle's motion, and also a probability $P$ which by Boltzmann formula reads $P = \exp(S/k)$. From this I was able to derive a number $a$ of results which may be found in reference [5].

The most important results are to me the following :

1) The Least Action Principle is only a particular case of the Second Principle of Thermodynamics.

2) The privilege, which Schrödinger has underlined as paradoxical, that the present Quantum mechanics grants to plane monochromatic waves and to stationary states of quantized systems is explained by the fact that these correspond to entropy maxima ; the other states do exist, but have much reduced probability.

### XI. On the necessary introduction of a random element in the double solution theory. The hidden thermostat and the Brownian motion of the particle in its wave

In the above arguments, it was assumed that the particle's motion in its wave was entirely determined by the guidance law. Hereafter will be shown why this standpoint cannot be wholly conserved.

To start with, Schrödinger's equation will be used as a good non-relativistic approximation. In §VI it was observed as a result of the continuity equation $(C)$, that the probability of finding the particle in a volume $d\tau$ is proportional to $a^2 d\tau$, $a$ being the $v$ wave's amplitude. Introducing the normed statistical wave, namely $\psi = C.v$, means that the considered probability's absolute value is $\mid \psi \mid^2$, a well known result. Difficulties arise however, when such considerations are made within the presently developed theory. This can be seen by examining a hydrogen atom in one of its $S$ type stationary states. The guidance formula $\vec{v} = -(\overrightarrow{\mathrm{grad}\,\phi})/m$ gives $v = 0$. The electron is then at rest in one point of the atom, and one fails to see how the continuity relation $(C)$ may justify the probability as $\mid \phi \mid^2 d\tau$. This leads to complementing the relation by introduction of a random element.

This difficulty looks the same as that encountered in classical statistical mechanics where Liouville's theorem, which yields a continuity formula in phase space, is not sufficient to establish that the probability for a representative point, a molecule in a gas, to be present in a volume element of its phase space, is proportional to this volume element. To justify this statement, one has to introduce in the molecule's motion, a random element which constantly perturbs the motion. Considering that this random element resulted from the collisions that each molecule undergoes with all the others in the gas, Boltzmann called it "molecular chaos".

By analogy, in the frame of double-solution theory, and in view of the well known fact that the probability for a particle to be in volume $d\tau$ is $\mid \psi \mid^2 d\tau$, a random element of hidden origin has to be admitted. This implies that the particle's regular motion, governed by the guidance law, is continuously submitted to random perturbations, with the result that the particle all the time switches from one guided trajectory to another. Taking these random perturbations into consideration, the continuity equation $\partial\rho/\partial t + \mathrm{div}\ \rho v = 0$ where $\rho = a^2$, and $v$ is the guidance velocity, justifies the probability law $\mid \psi \mid^2 = a^2$.

Finally, the particle's motion is the combination of a regular motion defined by the guidance formula, with a random motion of Brownian character. A simple comparison explains the possibility of such a superposition of motions. Consider a fluid's hydrodynamical flow. If placed on the surface of the fluid, a granule will move along with it. If this granule is massive enough, so that the action of collisions with the fluid's individual molecules has no visible effects, it will follow the lines of the hydrodynamical current flow, which may be compared with the guidance trajectories. But if the granule's mass is small enough, its motion will constantly be perturbed by individual collisions with the fluid's molecules. It will move according to both the regular motion following a current line of the general flow, and the Brownian motion, which will force it to switch constantly from one current line to another. An image is thus obtained of a random motion's superposition over the regular motion, similar to the one advanced for a particle.

In the above hydrodynamical comparison, the ensemble of all invisible molecules does play the part of the hidden thermostat. This latter by its continued interaction with the granule gives it a Brownian motion according to a well known concept of statistical thermodynamics. However, in the case of a particle which does not appear as subjected

to perturbations, such as an electron in a hydrogen atom, what could be the origin of these assumed perturbations ? To answer this question, any particle, even isolated, has to be imagined as in continuous "energetic contact" with a hidden medium, which constitutes a concealed thermostat. This hypothesis was brought forward some fifteen years ago by Bohm and Vigier [6], who named this invisible thermostat the "sub-quantum medium". As a further assumption, the particle is considered as continuously exchanging energy and momentum with such a hidden thermostat. These exchanges would happen regularly, in a well defined manner, if the guided motion existed alone, but a random energy exchange is superposed, which has a fluctuation character of well known kind in statistical thermodynamics.

If a hidden sub-quantum medium is assumed, knowledge of its nature would seem desirable. It certainly is of quite complex character. It could not serve as a universal reference medium, as this would be contrary to relativity theory. Moreover, it does not behave as a unique thermostat, but rather as an ensemble of thermostats, the temperatures of which are related to the proper energies $M_0 c^2$ of various kinds of molecules. Although interesting explanations have been proposed for this sub-quantum medium's nature, it seems premature to discuss the problem in the present paper.

## XII. Conclusion

Such is, in its main lines, the present state of the Wave mechanics interpretation by the double-solution theory, and its thermodynamical extension. I think that when this interpretation is further elaborated, extended, and eventually modified in some of its aspects, it will lead to a better understanding of the true coexistence of waves and particles about which actual Quantum mechanics only gives statistical information, often correct, but in my opinion incomplete.

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