Free falling electric charge in a static homogeneous gravitational field

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ABSTRACT. We reconsider the problem of a free falling electric charge in a static homogeneous gravitational field, specifically in a space-time domain in which the Riemann tensor vanishes and no electromagnetic field is present. We choose to describe the radiation emitted by the charge in terms of a general covariant quantity. We show that, under these assumptions, the charge, however accelerated, does not radiate, so that no contradiction arises with the Principle of Equivalence, which remains valid also for charged matter.

RÉSUMÉ. On reconsidère le problème d'une particule chargée en chute libre en un champ gravitationnel statique et homogène, spécifiquement dans un domaine d'espace-temps où le tenseur de Riemann s'annule et en absence de champs éléctromagnétiques. On choisit de décrire la radiation émise par la charge en termes d'une quantité covariante en sens général. On montre que, dans ces hypothèses, la charge, n'importe comment accélérée, n'emit pas de radiation, de façon qu'on ne relève aucune contradiction avec le principe d'equivalence, qui garde sa validité en présence de matière chargée.

1 Introduction

Einstein's Principle of Equivalence does not draw any explicit distinction between charged and neutral matter, whereas charged and neutral particles behave quite differently according to the Lorentz-Dirac equation. Since both the Principle of Equivalence and the Lorentz-Dirac equation describe the motion of a particle, but they do it in such a different way, a contradiction seems to arise.

More specifically, consider an electric charge in free fall in a static homogeneous gravitational field: an observer at rest with respect to the matter generating the gravitational field sees the charge in an accelerated motion (with respect to a system of reference which may be supposed to be inertial in the traditional sense) and so he would be tempted to conclude that the charge radiates; from the point of view of a free falling observer, the charge is at rest in a locally inertial reference frame, so that, on the basis of the Principle of Equivalence, he would instead conclude that the charge does not radiate.

The purpose of the present work is to give another contribution to the solution of this (apparent) contradiction.

In Sec. 1 we briefly rediscuss the Principle of Equivalence and the Lorentz-Dirac equation of motion. In Sec. 2 we recall the latest and, in our view, more interesting attempts at solving the problem, trying to explain why we consider them not yet fully satisfactory, while, at the same time, outlining the framework and the main lines of our approach. The calculations necessary to obtain the solution of our problem are developed in Sec. 3, where we also re-examine the scope of the Principle of Equivalence in the light of the conclusion reached about the question at hand.

2 The Principle of Equivalence

Consider a torsion-less space-time, so that (via Weyl's theorem) it is always possible to associate a locally inertial system of reference to any space-time point.

Principle of Equivalence 1 *A free falling system of reference in a gravitational field is a locally inertial system of reference, that is, the laws of physics are valid in it in the form established by Special Relativity.*

We shall need to analyse the problem in an extended space-time region. We therefore need considering a formal set-up in which global and local statements are equivalent. Since the connection is the mathematical tool that describes accelerations, and thereby gives information about inertiality, the above demand translates into the following mathematical assumption: if a system of reference is found such that the connection vanishes at a point, then it vanishes everywhere in that system.

There exists a theorem that translates the above assumption to an equivalent one, but in a form for which is used a tensorial language: in this formulation, one can say the Riemann tensor vanishes everywhere.

$$R^{\mu}_{\alpha\beta\rho} = 0 \tag{1}$$

A gravitational field for which the Riemann tensor is null is called a static homogeneous gravitational field [1].

With this assumption, we may give the statement of the Principle of Equivalence in the following form:

Principle of Equivalence 2 *A free falling system of reference in a static homogeneous gravitational field is a (globally) inertial system of reference, that is, the laws of physics are valid in it in the form established by Special Relativity.*

This is the form of the Principle of Equivalence that will be considered hereafter.

3 Equation of motion

The most general equation of motion is the Lorentz-Dirac equation, that we can write for a Riemannian space in which the metric has signature $(1, -1, -1, -1)$ as follows

$$m\frac{\delta u^{\mu}}{\delta s} = eg^{\mu\alpha}F_{\alpha\beta}u^{\beta} + \frac{2}{3}e^{2}(\frac{\delta^{2}u^{\mu}}{\delta s^{2}} + u^{\mu}\frac{\delta u^{\alpha}}{\delta s}\frac{\delta u_{\alpha}}{\delta s}) + e^{2}T(R^{\mu}_{\alpha\beta\rho}) \qquad (2)$$

where $T(R^{\mu}_{\alpha\beta\rho})$, called *tail*, is related to the Riemann tensor and vanishes if and only if the Riemann tensor vanishes (see [2] and [3]).

Taking into account condition $R^{\mu}_{\alpha\beta\rho} = 0$, we shall deal with a Lorentz-Dirac equation of the form

$$m\frac{\delta u^{\mu}}{\delta s} = eg^{\mu\alpha}F_{\alpha\beta}u^{\beta} + \frac{2}{3}e^{2}(\frac{\delta^{2}u^{\mu}}{\delta s^{2}} + u^{\mu}\frac{\delta u^{\alpha}}{\delta s}\frac{\delta u_{\alpha}}{\delta s})$$

As we can see, there are two terms depending on the charge: the first describes the interaction between the charged particle and an external elecromagnetic field; the second describes the selfinteraction of the charge with its own electromagnetic field. As a consequence, the equation discriminates between motions of differently charged particles in the presence of an external electromagnetic field through the first term, but, due to the second one it may also possibly discriminate between them intrinsically.

In case a charge dependent motion should arise when a non vanishing external electromagnetic field is present, we would be unable to ascertain if the differences of the two motions are due to the intrinsic character or to the external action of the electromagnetic field. So we choose to work in a space in which there is no any external electromagnetic field

$$F_{\mu\nu} \equiv 0 \tag{3}$$

so that the Lorentz-Dirac equation reduce to

$$\frac{\delta u^\mu}{\delta s} = \frac{2e^2}{3m}\left(\frac{\delta^2 u^\mu}{\delta s^2} + u^\mu \frac{\delta u^\alpha}{\delta s}\frac{\delta u_\alpha}{\delta s}\right) \tag{4}$$

The is the form of the Lorentz-Dirac equation of motion which shall be used hereafter.

4 The problem of free falling electric charge in a static homogeneous gravitational field

Suppose one tried to tackle the problem of a free falling electric charge in a static homogeneous gravitational field on the basis of the following hypotheses:

a) Radiation emitted by a charge should be treated as a covariant quantity, that is it should be described in tensorial terms

b) Principle of Equivalence is valid also for charged particles

c) *In general* (deliberatly vague), any accelerated charge emits radiation

Analyse the case of free falling electric charge in a static homogeneous gravitational field according to these hypotheses. Since, as we said, from the point of view of a comoving observer, the charge does not radiate, the tensor describing the emitted radiation should vanish in the free falling system; since, on the other hand, from the point of view of an observer at rest with respect to the matter generating the gravitational field, the charge should radiate, the tensor describing the emitted radiation should not vanish in that system. The contradiction can be avoided only dropping (at least) one of the three hypotheses, that is:

a) Consider the emitted radiation as a physical quantity depending on the system of reference

b) Consider the Principle of Equivalence valid only for the mechanics

c) Consider that, in some situation, even an accelerated charge may not radiate

In the past, many attempts at solving the problem have been made.

Some of them tried to solve the problem considering a theory in which non tensorial quantities were used (see, for example, [1]). In these attempts, in fact, one defined the emitted radiation as

$$I_0 = \frac{2}{3} e^2 \frac{du^\mu}{ds} \frac{du^\nu}{ds} g_{\mu\nu} \tag{5}$$

which is a tensor only underclosest linear transformations of coordinates but *not* a general covariant tensor, so that it is possible to find a system of reference in which it vanishes, even if it does not in another system of reference.

This is exactly what happens, namely one has $I_0 \neq 0$ in the system of reference at rest with respect to the matter, $I_0 = 0$ in the system of reference in free fall with the charge, and the two frames are connected by a non linear coordinate transformation, under which I_0 is not a tensor.

Even though in this way no contradiction arises, we do not find this attempt quite convincing. And this for two reasons: on the one hand, we deem unsatisfactory to deal with an emission of radiation in a not generally covariant term, and this when trying to solve an apparent contradiction arising in the framework of general relativity. For this reason, we will consider in this work

$$I = \frac{2}{3} e^2 \frac{\delta u^\alpha}{\delta s} \frac{\delta u_\alpha}{\delta s} \tag{6}$$

as the true expression for the emitted radiation. On the other hand, one is faced by another apparent contradiction, namely that arising from the circumstance that energy carried by electromagnetic radiation should be revealed or not by an array of counters depending on its state of motion with respect to the charge.

A different and more organic approach was considered first by M.Born in 1909, then by D.L.Drukey and finally by M.Bondi and T.Gold in the fifties and sixties; in this new approach the charge was not left to itself, its free fall being checked by an external electromagnetic field. The question whether the charge radiates was then given an affermative answer ([4], [5] and [6]).

The consequent contradiction about the validity of the Principle of Equivalence was removed in this manner: on one hand there is emission of radiation but on the other hand the Riemann tensor does not really vanish everywhere, but only locally. So, we can separate the space into two domains: in an immediate neighborhood of the charge, the Riemann tensor vanishes and one has no radiation; far from the charge, one has radiation and the Riemann tensor exhibits a curvature of the space-time: the Principle of Equivalence does not apply far from the charge, while near to it the Principle of Equivalence does not lead to any contradiction.

We do not find fully satisfactory this approach either: the necessity of considering an electromagnetic field appears to leave quite open the question concerning a free falling charge, an object which, after all, seems to be entitled to exist.

Our choice to consider a charge in *actual* free fall ($F_{\mu\nu} = 0$) in a *static homogeneous gravitational field* ($R^{\mu}_{\alpha\beta\rho} = 0$) is somewhat complementary to the last described.

5 Solving the problem

As we have just said, we shall take into account the conditions

$$F_{\mu\nu} \equiv 0 \tag{7}$$

$$R^{\mu}_{\alpha\beta\rho} = 0 \tag{8}$$

and the situation of a free falling electric charge in a static homogeneous gravitational field.

These assumptions leave us with a Lorentz-Dirac equation of motion of the form

$$\frac{\delta u^{\mu}}{\delta s} = \frac{2e^2}{3m}\left(\frac{\delta^2 u^{\mu}}{\delta s^2} + u^{\mu}\frac{\delta u^{\alpha}}{\delta s}\frac{\delta u_{\alpha}}{\delta s}\right) \tag{9}$$

The last equation must be supplied with the two sets of initial conditions concerning the four-position and the four-velocity; and, since one deals with a third order differential equation, the two sets must be supplemented by a third set concerning the four-acceleration. The three sets of initial conditions are

$$t(s = 0) = 0$$
$$x(s = 0) = x_0$$
$$y(s = 0) = 0$$
$$z(s = 0) = 0$$

and

$$\begin{aligned} u^t(s=0) &= 1 \\ u^x(s=0) &= 0 \\ u^y(s=0) &= 0 \\ u^z(s=0) &= 0 \end{aligned}$$

and, finally

$$\begin{aligned} \omega^t(s=0) &= 0 \\ \omega^x(s=0) &= 0 \\ \omega^y(s=0) &= 0 \\ \omega^z(s=0) &= 0 \end{aligned}$$

Since, according with the condition $R^{\mu}_{\alpha\beta\rho} = 0$, the space is flat, there exists an overall inertial system, in which one can write the Lorentz-Dirac equation of motion in the form established by Special Relativity, that is

$$\omega^{\mu} = \frac{2e^2}{3m}\left(\frac{d\omega^{\mu}}{ds} + u^{\mu}\omega^{\alpha}\omega_{\alpha}\right) \tag{10}$$

In a static homogeneous gravitational field the free motion is characterized by the condition

$$\omega^{\alpha}\omega_{\alpha} = -g^2 \tag{11}$$

with g constant. Taking the first derivatives with respect to s one has

$$\frac{d\omega^{\alpha}}{ds}\omega_{\alpha} = 0$$

and taking the contraction between the Lorentz-Dirac equation and the acceleration and simplifying one obtains

$$-g^2 = \frac{2e^2}{3m}\frac{d\omega^{\mu}}{ds}\omega_{\mu} = 0$$

so that

$$\omega^{\alpha}\omega_{\alpha} = 0 \tag{12}$$

This result implies

$$I = 0 \tag{13}$$

in the inertial system of reference, but, since it *is* a tensor, the result holds in all the systems of reference. This means that no radiation is emitted by the charge.

We next show that the motion is a geodesic motion. The motion of the partcle is characterized by the condition

$$\frac{\delta u^\alpha}{\delta s}\frac{\delta u_\alpha}{\delta s}=0 \tag{14}$$

and the Lorentz-Dirac equation reduces to

$$\frac{\delta u^\mu}{\delta s}=\frac{2e^2}{3m}\frac{\delta^2 u^\mu}{\delta s^2} \tag{15}$$

so that the equations for the single components have been separated. Their solutions are

$$u^\mu(s)=u^\mu(0)+\frac{2e^2\omega^\mu(0)}{3m}[\exp(\frac{3m}{2e^2}s)-1]$$

which, taking into account the initial conditions, become

$$u^\mu(s)=u^\mu(0)$$

hence

$$\omega^\mu(s)\equiv 0 \tag{16}$$

in the inertial system of reference. In this system the connection is also vanishing. Since, in general,

$$\frac{\delta u^\mu}{\delta s}=\frac{du^\mu}{ds}+\Gamma^\mu_{\alpha\beta}u^\alpha u^\beta$$

one obtains

$$\frac{\delta u^\mu}{\delta s}\equiv 0 \tag{17}$$

and, since this *is* a tensor, the previous relation holds in every system of reference.

The solution of the Lorentz-Dirac equation in our situation is a space-time geodesic. This result is the same one obtain considering a neutral particle, so that there is no distinction between the two kinds of objects.

One further work on this problem is due to Rohrlich and Fulton [7]; in their paper, they give a proof of the fact that the right side of the Lorentz-Dirac equation vanishes, namely, writing the Lorentz-Dirac equation of motion in a flat space as

$$m\frac{\delta u^\mu}{\delta s} = eg^{\mu\alpha}F_{\alpha\beta}u^\beta + \Gamma^\mu \tag{18}$$

they show that

$$\Gamma^\mu \equiv 0 \tag{19}$$

But the term that describes the radiation is decomposable into two terms; hence, even if the term describing the radiation vanishes, the single terms in which it is decomposed may not. In our situation, we have also required the absence of the external electromagnetic field. In this case, if the term describing the radiation vanishes then the single terms vanish too and the motion is once again a geodesic.

6 Conclusions

To conclude: we considered two observers: the observer at rest with respect to the matter source of the gravitational field sees the charge perform the hyperbolic motion, typical of relativistic free fall, hence not radiating; the observer in free fall with the charge keeps seeing it at rest, henceagain not radiating, as required by covariance and without any contradiction.

If the Principle of Equivalence is not valid for all of electromagnetism, then a proof for this fact cannot be obtained by this kind of an experiment.

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