Geometry and Quantum Mechanics

B.G. Sidharth

B.M. Birla Science Centre, Adarshnagar, Hyderabad - 500 063, India

Attempts for a geometrical interpretation of Quantum Theory were made, notably the deBroglie-Bohm formulation. This was further refined by Santamato who invoked Weyl's geometry. However these attempts left a number of unanswered questions. In the present paper we return to these two formulations, in the context of recent studies invoking fuzzy spacetime and noncommutative geometry. We argue that it is now possible to explain the earlier shortcomings. At the same time we get an insight into the geometric origin of the deBroglie wavelength itself as also the Wilson-Sommerfeld Quantization rule.

1 Introduction

One of the fruitful approaches to Quantum Mechanics was the so called deBroglie-Bohm hydrodynamical formulation [1], which originated with Madelung and was developed by Bohm using deBroglie's pilot wave ideas. In this formulation, while the initial position coordinates in a Quantum Mechanical trajectory, are random, the trajectories themselves are determined by classical mechanics. Quantum Mechanics enters through an inexplicable Quantum potential which is again related to the wave function. This has been a stumbling block in the acceptance of the formulation.

Much later, Santamato further developed the deBroglie-Bohm formulation by relating the mysterious Quantum potential to fundamental geometric properties, by invoking Weyl's geometry [2, 3, 4]. The net result was that the mysterious Quantum effects were shown to be related to the geometric structure of space specifically to the curvature. Unfortunately, Weyl's theory itself did not find favour [5]. Apart from anything else, the theory sought to unify electromagnetism with gravitation, but

on closer scrutiny, in this geometrical structure the two interactions were actually independent and ad hoc entities as noted by Einstein himself [6].
We will now reexamine all this in the light of recent developments in fuzzy spacetime and noncommutative geometry, and argue that infact, once the underlying fuzzyness is recognized, then the above apparent difficulties disappear.

2 The deBroglie-Bohm Formulation and Extensions

Let us briefly review the above theory [7]. We start with the Schrodinger equation

$$\imath\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\psi + V\psi \tag{1}$$

In (1), the substitution

$$\psi = Re^{\imath S/\hbar} \tag{2}$$

where R and S are real functions of $\vec{r}$ and t, leads to,

$$\frac{\partial\rho}{\partial t} + \vec{\nabla}\cdot(\rho\vec{v}) = 0 \tag{3}$$

$$\frac{1}{\hbar}\frac{\partial S}{\partial t} + \frac{1}{2m}(\vec{\nabla}S)^2 + \frac{V}{\hbar^2} - \frac{1}{2m}\frac{\nabla^2 R}{R} = 0 \tag{4}$$

where

$$\rho = R^2, \vec{v} = \frac{\hbar}{m}\vec{\nabla}S$$

and

$$Q \equiv -\frac{\hbar^2}{2m}(\nabla^2 R/R) \tag{5}$$

Using the theory of fluid flow, it is well known that (3) and (4) lead to the Bohm alternative formulation of Quantum Mechanics. In this theory there is a hidden variable namely the definite value of position while the so called Bohm potential Q can be non local, two features which do not find favour with physicists.

Let us now briefly review Weyl's ideas. He postulated that in addition to the general coordinate transformations of General Relativity, there were also gauge transformations which multiplied all components of the metric tensor $g_{\mu\nu}$, by an arbitrary function of the coordinates. So, the line elements would no longer be invariant. In its modern version, the metric tensor is normalized so that its determinant is given by [5],

$$|g_{\mu\nu}| = -1,$$

while it now transforms as a tensor density of weight minus half, and not as a tensor. This leads to the circumstance that there is now a covariant derivative involving an arbitrary function of coordinates Φ_μ given by

$$\Phi_\sigma = \Gamma^{\rho}_{\rho\sigma}, \tag{6}$$

where the Γ's denote the Christofell symbols. Weyl identified Φ_μ in (6) with the electromagnetic potential. It must be noted that in Weyl's geometry, even in a Euclidean space there is a covariant derivative and a non vanishing curvature R.
Santamato exploits this latter fact, within the context of the deBroglie-Bohm theory and postulates a Lagrangian given by

$$L(q, \dot{q}, t) = L_c(q, \dot{q}, t) + \gamma(\hbar^2/m)R(q, t),$$

He then goes on to obtain the equations of motion like (1),(2), etc. by invoking an Averaged Least Action Principle

$$\begin{aligned} I(t_0, t_1) = E\left\{\int_{t_0}^{t} L^*(q(t,\omega), \dot{q}(t,\omega), t)dt\right\} \\ = \text{minimum}, \end{aligned} \tag{7}$$

with respect to the class of all Weyl geometries of space with fixed metric tensor. This now leads to the Hamilton-Jacobi equation

$$\partial_t S + H_c(q, \nabla S, t) - \gamma(\hbar^2/m)R = 0, \tag{8}$$

and thence to the Schrodinger equation (in curvi-linear coordinates)

$$\begin{aligned} \imath\hbar\partial_t\psi = (1/2m)\left\{[(\imath\hbar/\sqrt{g})\partial_\imath\sqrt{g}A_\imath]g^{\imath k}(\imath\hbar\partial_k + A_k)\right\}\psi \\ + [V - \gamma(\hbar^2/m)\dot{R}]\psi = 0, \end{aligned} \tag{9}$$

As can be seen from the above, the Quantum potential Q is now given in terms of the scalar curvature R.
We would now like to relate the arbitrary functions Φ of Weyl's formulation with a noncommutative spacetime geometry, as was shown recently. Let us write the product $dx^\mu dx^\nu$ of the line element,

$$ds^2 = g_{\mu\nu} d dx^\mu dx^\nu,$$

as a sum of half its anti-symmetric part and half the symmetric part. The line element now becomes $(h_{\mu\nu} + \bar{h}_{\mu\nu}) dx^\mu dx^\nu$. This leads us back to the Weyl geometry because the metric tensor $\bar{h}_{\mu\nu}$ now becomes a tensor density [8, 9].
In other words it is the underlying fuzzyness of space time as expressed by

$$[dx^\mu, dx^\nu] \approx l^2 \neq 0 \tag{10}$$

l being a typical length scale $\sim 0(dx^\mu)$, that brings out Weyl's geometry, not as an ad hoc feature, but as a truly geometrical consequence, and therefore also legitimises Santamato's postulative approach of extending the deBroglie-Bohm formulation.
At an even more fundamental level, this formalism gives us the rationale for the deBroglie wave length itself. Because of the noncommutative geometry in (10) space becomes multiply connected, in the sense that a closed circuit cannot be shrunk to a point within the interval. Let us consider the simplest case of double connectivity. In this case, if the interval is of length λ, we will have, using (5),

$$\Gamma \equiv \int_c m\vec{V} \cdot d\vec{r} = h \int_c \vec{\nabla} S \cdot d\vec{r} = h \oint dS = mV\pi\lambda = \pi h \tag{11}$$

whence

$$\lambda = \frac{h}{mV} \tag{12}$$

In (11), the circuit integral was over a circle of diameter λ. Equation (12) shows the emergence of the deBroglie wavelength. This follows from the noncommutative geometry of space time, rather than the physical Heisenberg Uncertainty Principle. Remembering that Γ in (11) stands for the angular momentum, this is also the origin of the Wilson-Sommerfeld quantization rule, an otherwise mysterious Quantum Mechanical prescription.

3 Discussion

1. We would like to stress that Santamato's treatment via Weyl's geometry, of the deBroglie-Bohm formulation was postulative (Cf. equations (7), (8), (9)), while the Weyl formulation itself had not found favour for its original motivation. Perhaps this was the reason why Santamato's formulation was not taken so seriously. On the other hand, we have argued from the point of view of the noncommutative geometry (10), which, after many decades, is now coming to be recognized in the context of Quantum Superstring theory and Quantum Gravity.
2. It is well known that the so called Nelsonian stochastic process resembles the deBroglie-Bohm formulation, with very similar equations [10, 11]. However in this former case, both the position and velocity are not deterministic because of an underlying Brownian process. In this formulation the diffusion constant of the theory has to be identified with,

$$\nu = \frac{h}{m}$$

These are the extra features in this stochastic formulation, rather than the Quantum potential, which also appears in the equations. It has been shown by the author [12, 7], that both the similar approaches infact can be unified for relativistic velocities, by considering quantized vortices originating from (11)of the order of the deBroglie, now the Compton scale l. This immediately brings us back to the fuzzy noncommutative geometry (10). At the same time it must be pointed out that the supposedly unsatisfactory non local features of the Quantum potential Q become meaningful in the above context at the Compton scale, within which indeed we have exactly such non local effects [13].
It may be pointed out that more recently we have been led back to the background Quantum vaccuum and the underlying Zero Point Field, now christened dark energy by the observation of the cosmological constant implied by the accelerated expansion of the universe [7] and it is this ZPF which provides the Brownian process of the stochastic theory. As pointed out by Nottale [14], such a Brownian process also eliminates the hidden variable feature of the deBroglie- Bohm formulation.
Interestingly it has been argued by Enz [15] that a particle extension, as is implied in the above considerations in the form of a quantized vortex or fuzzy space time, is the bridge between the particle and wave aspects. Infact at this scale, there is zitterbewegung, reminiscent of deBroglie's

picture of a particle as an oscillator. Originally Dirac had interpreted the zitterbewegung oscillations as unphysical, which are removed by the fact that due to the Uncertainty Principle we cannot go down to arbitrarily small space time intervals, so that our space time points are averages over Compton scale intervals (Cf.[7, 16] for a discussion).

3. It is also interesting to note that Santamato's tying up of the Quantum potential with the curvature R has been interpreted as being the result of cosmic fluctuations [17], this being a special case of a more general but identical earlier result of the author [18, 19].

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